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Motion of a spherical microcapsule freely suspended in a linear shear flow

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The motion of a spherical microcapsule freely suspended in a simple shear flow is studied. The particle consists of a thin elastic spherical membrane enclosing an incompressible Newtonian viscous fluid. The motions of the internal liquid and of the suspending fluid are both described by Stokes equations. On the deformed surface of the membrane, continuity of velocities is imposed together with dynamic equilibrium of viscous and elastic forces. Since this problem is highly nonlinear, a regular perturbation solution is sought in the limiting case where the deviation from sphericity is small. In particular, the nonlinear theory of large deformation of membrane shells is expanded up to second-order terms. The deformation and orientation of the microcapsule are obtained explicitly in terms of the magnitude of the shear rate, the elastic coefficients of the membrane, the ratio of internal to external viscosities. It appears that the very viscous capsules are tilted towards the streamlines, whereas the less viscous particles are oriented at nearly 45° to the streamlines. The tank-treading motion of the membrane around the liquid contents is predicted by the model and appears as the consequence of a solid-body rotation superimposed upon a constant elastic deformation.

1. Introduction

Whenever a deformable solid is subjected to viscous forces due to the flow of a fluid, it will react by altering its shape. Similarly, the very presence of the solid modifies the flow field until, it is hoped, a steady state is attained where the solid reaches a steady configuration, generally different from its original one. Such problems are usually highly nonlinear in essence, but there are some particular situations where a solution can be obtained. This is the case when the solid is so rigid that its deformation can be neglected in a first approximation, at least as regards the formulation of boundary conditions at the fluid–solid interface. This is the approach taken for example to study the effect of a flowing fluid, such as wind or water, on structures. Another case deals with large deformations or at least large displacements of the solid under the influence of an isotropic pressure (e.g. inflation or deflation of domes). In this situation, the load on the solid is particularly simple, and no viscous shearing forces are considered. The general case, involving large deformations of a solid under a general flow of a fluid, is very difficult to formulate and also to solve. One of the reasons for this is due to the fact that two different reference systems are needed: an Eulerian reference system for the fluid problem, and a Lagrangian reference system for the solid mechanics problem. If, furthermore, the solid is allowed to move freely under the influence of the fluid, then the switch between the Lagrangian and the Eulerian representations becomes very complex indeed. Another common feature of such problems consists in the fact that most boundary conditions are imposed at the interface between the fluid and the solid, whose position is unknown. Finally, the theory of large elastic deformations must generally be used. Unfortunately, owing to the complexity of this theory, the existing analytical solutions are very few, and most of them have been derived for geometrically simple bodies, under simple loads.

Despite the difficulties of such problems, there have been a number of attempts to find a solution in some particular cases, such as the behaviour of homogeneous elastic solids. The case of elastic spheres suspended in a linear shear flow has been treated by Roscoe (1967) and by Goddard & Miller (1967) while the deformation of elastic spheroids in Couette flow has been studied by Lingard & Whitmore (1974). By making use of lubrication theory, Lighthill (1968) and later Fitzgerald (1969) have determined the motion of elastic pellets through capillary tubes. Another class of such problems deals with the behaviour of liquid droplets freely suspended in another liquid. In this case, studied by Taylor (1932), Chaffey & Brenner (1967), Frankel & Acrivos (1970), Barthès-Biesel & Acrivos (1973) amongst others, the particle is a liquid continuum, and the interface between the two liquids can be viewed as a very particular membrane, infinitely shearable, but obeying Laplace's law as regards normal stresses.

The object of this paper is to study the behaviour of special types of particles to be termed microcapsules. They consist of a thin elastic solid skin, enclosing a Newtonian incompressible liquid. Such particles may be thought of as models of red blood cells, although the exact rheology of the red cell membrane is still open for discussion. They are also encountered in emulsions stabilized by interfacial cross-linking polymerization. The motion of ellipsoidal microcapsules suspended in a simple shear flow of another fluid has been considered by Richardson (1974), as a model of red blood cells suspensions. The linear theory of membrane deformation was used, although it is apparent from the results that the deformed shape of the particle is far from ellipsoidal, and that the flow field should be computed again. Also the influence of the internal fluid was neglected.

Here, the motion of spherical microcapsules, suspended in a simple shear flow, is studied. The membrane of the cell has arbitrary properties, and can achieve large deformations. However, owing to the particular geometry chosen, an analytical asymptotic solution can be obtained. As a result, the solution shows how the deformation and the orientation of the particle depend upon its physical properties. In spite of the restrictive assumption made on the geometry, this model nevertheless leads to qualitatively interesting results, that can be used as a first approximation in the study of the deformability of red blood cells.

First we describe the problem and give the equations of motion and the boundary conditions for the fluids. In § 3 we review briefly the theory of large deformations of thin elastic membranes. The regular perturbation technique used to solve the problem is explained in the fourth section, while the two first approximations to the solution are given in the last sections.

2. Description of the problem

The problem consists in determining the motion of a spherical microcapsule, when it is freely suspended in a simple shear flow. The particle's boundary is a thin spherical membrane, of diameter $2d$, and of constant thickness h in its stress-free configuration.

The assumption of thinness implies that $h/d \ll 1$, and that the membrane can be essentially treated as a two-dimensional surface. The material of the membrane is assumed to be isotropic, incompressible and to have arbitrary elastic properties, characterized by a general elastic modulus E. Consequently the membrane is treated as the two-dimensional limit of a three-dimensional solid, instead of as a purely two-dimensional medium, which would be more general. The particle is filled with an incompressible Newtonian fluid of viscosity $\lambda\mu$. It is freely suspended in another Newtonian incompressible fluid of viscosity μ. The suspension is then subjected to a simple shear flow of magnitude G. Under the influence of the viscous shearing forces applied on the particle membrane, the latter deforms until a steady shape is reached. This is obtained when, at every point of the membrane, the viscous forces are in dynamic equilibrium with the elastic tensions generated by the deformation. Consequently, the motion of the fluids and the deformation of the membrane are linked and must be solved for simultaneously. This leads to the definition of two reference frames. The fluid problem is referred to an Eulerian frame, $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$, moving with the centre of mass of the particle. The co-ordinates of a point in this frame are denoted x_i or X_i. The membrane problem is referred to a Lagrangian frame ($\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{n}$ before deformation; $\mathbf{A}_1$, $\mathbf{A}_2$, $\mathbf{N}$ after deformation), corresponding to local curvilinear co-ordinates, y^i. The two frames are shown on figure 1.

All quantities are first non-dimensionalized: lengths by d, velocities by Gd, stresses in the fluids by μG, tensions in the membrane by Eh. The Reynolds number of the flow, based on the particle dimensions, $\rho G d^2/\mu$, is assumed to be much smaller than unity, so that inertial effects can be neglected. Consequently the motion of the fluids is described by the Stokes equations

$$\frac{\partial^2 v_i}{\partial x_k \partial x_k} = \frac{\partial p}{\partial x_i}, \quad \frac{\partial v_i}{\partial x_i} = 0 \quad \text{for} \quad r \geqslant f; \tag{2.1}$$

$$\lambda \frac{\partial^2 v_i^*}{\partial x_k \partial x_k} = \frac{\partial p^*}{\partial x_i}, \quad \frac{\partial v_i^*}{\partial x_i} = 0 \quad \text{for} \quad r < f; \tag{2.2}$$

$$r = (x_i x_i)^{\frac{1}{2}}.$$

Latin indices represent the values 1, 2, or 3, while Greek indices will only take the values 1 and 2. Einstein's summation convention over repeated indices is used. The velocity and the pressure in the fluid, at position x_i, are respectively denoted v_i and p. Starred quantities refer to the internal liquid. The equation of the deformed surface of the particle is given by

$$r = f\left(\frac{x_1}{r}, \frac{x_2}{r}, \frac{x_3}{r}\right).$$

The boundary condition far from the particle is

$$v_i \to e_{ik} x_k + \Omega_{ik} x_k \quad \text{as} \quad r \to \infty, \tag{2.3}$$

where, for a simple shear flow,

$$e_{12} = e_{21} = \Omega_{12} = -\Omega_{21} = \tfrac{1}{2},$$

all other components being zero. On the deformed surface of the particle, continuity of velocities is required,

$$v_i = v_i^* = v_i^{(m)} \quad \text{at} \quad r = f, \tag{2.4}$$

$$v_i N_i = 0, \tag{2.5}$$

where $v_i^{(m)}$ is the membrane velocity and N_i the outer normal unit vector to the surface.

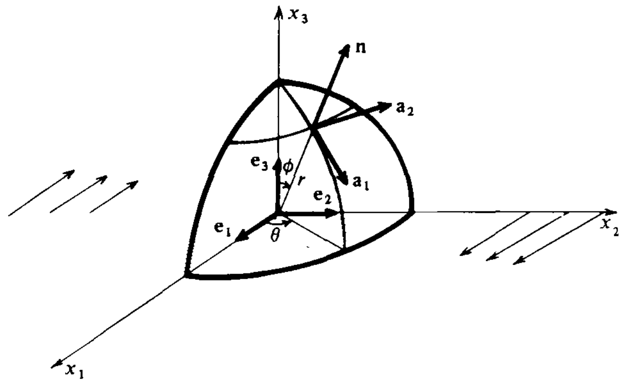

FIGURE 1. The co-ordinate systems. The flow problem is referred to the $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ frame, moving with the centre of mass of the particle. The membrane problem is referred to a local curvilinear frame, $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{n}$, corresponding to spherical co-ordinates θ, ϕ, r.

The final boundary condition requires that the membrane be in equilibrium under the load due to viscous stresses σ_{ij} and σ^*_{ij} in the two fluids. Thus at each point of the membrane the components in the $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ frame of the resultant force per unit area are given by

$$p_i = (\sigma_{ij} - \sigma^*_{ij}) N_j. \tag{2.6}$$

Owing to the assumption of thinness, the variations of the elastic stresses across the membrane are neglected, so that these stresses, averaged over the thickness h, give rise to tensions shown on figure 2. The equilibrium equations are written in the Lagrangian reference frame linked to every point of the deformed surface, to be precisely defined in the following section. With respect to these axes, the tensions have contravariant components $\tau^{\alpha\beta}$, whereas the components of the load are denoted q^α and q^3. Then, neglecting the inertia of the membrane, the condition of equilibrium can be written as

$$\tau^{\alpha\beta}|_\alpha + \frac{\mu G d}{Eh} q^\beta = 0, \tag{2.7}$$

$$\tau^{\alpha\beta} B_{\alpha\beta} + \frac{\mu G d}{Eh} q^3 = 0, \tag{2.8}$$

where $B_{\alpha\beta}$ is the second fundamental form of the deformed surface and where the bar denotes covariant differentiation, defined in terms of the Christoffel symbols $\Gamma^\gamma_{\alpha\beta}$ of the deformed middle surface of the membrane by

$$\tau^{\alpha\beta}|_\alpha = \frac{\partial \tau^{\alpha\beta}}{\partial y^\alpha} + \Gamma^\alpha_{\alpha\gamma} \tau^{\gamma\beta} + \Gamma^\beta_{\alpha\gamma} \tau_{\alpha\gamma}. \tag{2.9}$$

There remains now to relate the stresses in the membrane to the displacements of its material points. This is the aim of the theory of finite deformations of elastic membranes, which is summarized in the following section.

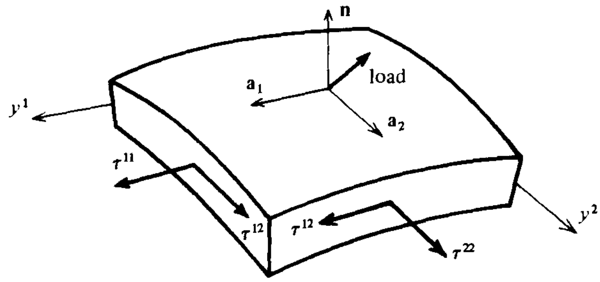

FIGURE 2. Forces acting on a membrane element.

3. Finite deformations of elastic membranes

The general theory of large elastic deformations and its application to shells, has been treated in considerable detail in two books, Green & Zerna (1954) and Green & Adkins (1960). In the case of thin membranes, this theory has been reformulated by Corneliussen & Shield (1961) whose notation we shall use here.

The main assumptions of shell theory, known as Kirchoff's hypotheses, are the following. First the shell is thin, so that its behaviour can be reduced to that of its middle surface. Second, the stresses in the direction normal to the middle surface are neglected. Finally, elements normal to the surface before deformation remain normal to the surface after deformation. Then, instead of stresses in the shell body, stress resultants are considered. The problem can be markedly simplified by making the so-called membrane approximation, which consists in neglecting stress couples and shearing forces normal to the membrane with respect to stress resultants in the membrane plane. This is equivalent to neglecting the bending resistance of the membrane material.

First, the metric of the middle surface in the deformed state must be specified. Non-dimensional quantities are used throughout. Let y^1, y^2 be general curvilinear co-ordinates of a point on the middle surface of the shell. The position vector of this point is $\mathbf{a}(y^1, y^2)$ before deformation and $\mathbf{A}(y^1, y^2)$ after deformation. The position vector of any point of the membrane is given by

$$\left.\begin{aligned} &\mathbf{x} = \mathbf{a}(y^1, y^2) + y^3\mathbf{n} \quad \text{in the undeformed state,} \\ &\mathbf{X} = \mathbf{A}(y^1, y^2) + k(y^1, y^2)\, y^3\mathbf{N} \quad \text{in the deformed state,} \\ &\mathbf{u} = \mathbf{X} - \mathbf{x} \quad \text{is the displacement of each point,} \end{aligned}\right\} \tag{3.1}$$

with $|y^3| \leqslant h/2d$; $\mathbf{n}$ and $\mathbf{N}$ are the unit normal vectors to the middle surface before and after deformation, k is the ratio of the thickness of the shell after deformation to the thickness of the shell before deformation.

A local co-ordinate system is defined for each point, with covariant base vectors given by

$$\mathbf{a}_{,\alpha} = \frac{\partial \mathbf{a}}{\partial y^\alpha}, \quad \mathbf{n} = \frac{\mathbf{a}_{,1} \wedge \mathbf{a}_{,2}}{|\mathbf{a}_{,1} \wedge \mathbf{a}_{,2}|}; \tag{3.2}$$

$$\mathbf{A}_{,\alpha} = \frac{\partial \mathbf{A}}{\partial y^\alpha}, \quad \mathbf{N} = \frac{\mathbf{A}_{,1} \wedge \mathbf{A}_{,2}}{|\mathbf{A}_{,1} \wedge \mathbf{A}_{,2}|}. \tag{3.3}$$

We adopt here the classical notation, where a comma denotes differentiation with respect to y^α. The covariant metric tensors of the surface before and after deformation are then

$$a_{\alpha\beta} = \mathbf{a}_{,\alpha}.\mathbf{a}_{,\beta} \quad \text{and} \quad A_{\alpha\beta} = \mathbf{A}_{,\alpha}.\mathbf{A}_{,\beta}, \tag{3.4}$$

while their associated determinants are respectively denoted a and A. The contravariant components of these tensors are defined in the usual way, e.g.

$$A^{\alpha\beta} = \frac{C_{\alpha\beta}}{A}, \tag{3.5}$$

where $C_{\alpha\beta}$ is the cofactor of $A_{\alpha\beta}$ in the determinant A. The three invariants of the strain tensor may be defined as

$$I_1 = a^{\alpha\beta}A_{\alpha\beta}+k^2, \quad I_2 = I_3(a_{\alpha\beta}A^{\alpha\beta}+k^{-2}), \quad I_3 = k^2\frac{A}{a}, \tag{3.6}$$

with $I_3 = 1$ for an incompressible material.

The metric properties of the surface are further specified by two tensors, the Christoffel symbol given by

$$\Gamma^\alpha_{\beta\gamma} = \tfrac{1}{2}A^{\alpha\mu}[A_{\mu\beta,\gamma}+A_{\mu\gamma,\beta}-A_{\beta\gamma,\mu}] \tag{3.7}$$

and the second fundamental form defined as

$$B_{\alpha\beta} = \mathbf{N}.\mathbf{A}_{,\alpha,\beta}. \tag{3.8}$$

Now, the tensions in the membrane must be related to the deformations, by means of the rheological equation of state of the membrane material, which, in its general elastic form, is given by

$$\tau^{\alpha\beta} = kh[\Phi a^{\alpha\beta}+\Psi D^{\alpha\beta}+PA^{\alpha\beta}]/Eh, \tag{3.9}$$

where

$$D^{\alpha\beta} = k^2a^{\alpha\beta}+[a^{\delta\gamma}a^{\alpha\beta}-a^{\alpha\delta}a^{\beta\gamma}]A_{\gamma\delta} \tag{3.10}$$

and where the material functions Φ, Ψ and P are related to the strain energy function $W(I_1, I_2, I_3)$ of the material by

$$\Phi = \frac{2}{I_3^{\frac{1}{2}}}\frac{\partial W}{\partial I_1}; \quad \Psi = \frac{2}{I_3^{\frac{1}{2}}}\frac{\partial W}{\partial I_2}; \quad P = 2I_3^{\frac{1}{2}}\frac{\partial W}{\partial I_3}. \tag{3.11}$$

When the solid is incompressible, W is a function of I_1 and I_2 only, and thus P cannot be evaluated from (3.11). It is however non-zero, since it represents the value of $\partial W/\partial I_3$ for $I_3 = 1$. However, P can be eliminated from the problem by means of the membrane hypothesis, $\tau^{33} = 0$, which is expressed by

$$\Phi+\Psi(I_1-k^2)+Pk^{-2} = 0. \tag{3.12}$$

The analytical expression of W depends on the rheological properties of the material. For an incompressible elastic solid, a commonly used form of W is the so-called Mooney equation

$$W = C_1(I_1-3)+C_2(I_2-3), \tag{3.13}$$

where C_1 and C_2 are constants. The case $C_2 = 0$ corresponds to an incompressible neo-Hookean solid.

The problem is now completely defined by equations (2.1) to (2.9) and (3.1) to (3.12). It appears that the solution depends on two main dimensionless parameters, namely the viscosity ratio λ, and the ratio $\mu Gd/Eh$ of viscous deforming forces to elastic shape-restoring forces. As such, though, this set of equations is highly nonlinear, since it constantly involves the geometry of the deformed surface. Consequently, a solution is sought in the special case where the shape of the microcapsule remains nearly spherical, and where it is possible to develop a perturbation solution.

4. Perturbation solution of the equations

We assume that the ratio of viscous to elastic forces, $\mu Gd/Eh$, is much smaller than unity. This ensures that the deformations of the sphere remain small. Cox (1969) and Frankel & Acrivos (1970) have shown how to obtain a regular perturbation solution in a similar situation encountered for liquid droplets. Their method consists in expanding all quantities of interest, here velocities, fluid stresses, membrane tensions, membrane displacements, in terms of a small parameter ϵ:

$$\epsilon = \mu Gd/Eh, \quad \epsilon \ll 1.$$

The equation of the middle surface of the membrane of the capsule is described by

$$r(x_1, x_2, x_3) = 1 + \epsilon \overset{(1)}{f}(x_1, x_2, x_3) + \epsilon^2 \overset{(2)}{f}(x_1, x_2, x_3) + \ldots$$

while, for example, the velocity of the external fluid is given by

$$v_i = \overset{(0)}{v_i} + \epsilon \overset{(1)}{v_i} + \ldots.$$

The boundary conditions (2.4) and (2.5) are expanded in terms of ϵ, and all terms are evaluated for $r = 1$. Consequently, the $O(1)$ continuity of velocity condition, and the viscous load applied on the membrane respectively become

$$\overset{(0)}{v_i} = \overset{(0)}{v_i^*} = \overset{(0)}{v_i^{(m)}}, \tag{4.1}$$

$$\overset{(0)}{v_i} x_i = 0, \tag{4.2}$$

$$\overset{(0)}{p_i} = (\overset{(0)}{\sigma_{ij}} - \overset{(0)}{\sigma_{ij}^*})\, x_j. \tag{4.3}$$

Similarly, to $O(\epsilon)$, the velocity condition and the viscous load are given by

$$\overset{(1)}{v_i} + \overset{(1)}{f} x_j \frac{\partial \overset{(0)}{v_i}}{\partial x_j} = \overset{(1)}{v_i^*} + \overset{(1)}{f} x_j \frac{\partial \overset{(0)}{v_i^*}}{\partial x_j} = \overset{(1)}{v_i^{(m)}} + \overset{(1)}{f} x_j \frac{\partial \overset{(0)}{v_i^{(m)}}}{\partial x_j}, \tag{4.4}$$

$$\overset{(1)}{v_i} x_i + \overset{(1)}{f} x_j \frac{\partial \overset{(0)}{v_i}}{\partial x_j} x_i - \frac{\partial \overset{(1)}{f}}{\partial x_i} \overset{(0)}{v_i} = 0, \tag{4.5}$$

$$\overset{(1)}{p_i} = \left[\overset{(1)}{\sigma_{ij}} - \overset{(1)}{\sigma_{ij}^*} + \overset{(1)}{f} x_k \frac{\partial}{\partial x_k} (\overset{(0)}{\sigma_{ij}} - \overset{(0)}{\sigma_{ij}^*})\right] x_j - \frac{\partial \overset{(1)}{f}}{\partial x_j} (\overset{(0)}{\sigma_{ij}} - \overset{(0)}{\sigma_{ij}^*}). \tag{4.6}$$

These boundary conditions are valid for all points x_i of the interface, regardless of the initial position of the material point which is at x_i at time t.

Let us consider now the case where Eh becomes infinitely large, or ϵ infinitely small. Then, the membrane, being infinitely rigid, will remain spherical. The capsule will behave essentially as a solid sphere; that is, it will take a solid-body rotation corresponding to the fluid vorticity. Consequently the $O(1)$ terms in the expansions of the internal stress and of the membrane velocity can be evaluated immediately:

$$\left.\begin{aligned} \sigma^*_{ij} &= \overset{(0)}{p}{}^* \delta_{ij} + \epsilon \overset{(1)}{\sigma}{}^*_{ij} + \dots, \\ v^{(m)}_i &= \Omega_{il} x_l + \epsilon \overset{(1)}{v}{}^{(m)}_i + \dots \end{aligned}\right\} \tag{4.7}$$

The method of solution is now clear. The first term in the expansion of the flow field corresponds to the shear flow around a freely rotating sphere. The stress force on each point of the membrane can be easily computed. However, the equations of equilibrium of the membrane (2.7) and (2.8) are expressed for steady state and for negligible membrane inertia. But, since the material of the membrane is uniform, the identity of the material point passing through position x_i at a given time is irrelevant. Consequently the rotation of the membrane can be ignored when solving the equilibrium equation, and the displacement of each point is computed in the usual fashion. From the expression of this displacement, it is then possible to determine the new shape of the capsule, and to compute the new membrane velocity by superposing the rotational motion. Then, the velocity being specified everywhere on the boundary of the domain, the next-order approximation for the flow field is computed and the process is repeated. It appears now that the perturbation technique results in decoupling the fluid and membrane problems, which are solved alternately.

In order to complete the description of the perturbation procedure, there remains now to show how the theory of finite deformation of elastic membranes can be expanded. The method has been outlined by Lomen (1964) for a different problem, namely the superposition of a small deformation upon a finite deformation. This type of problem is often linked to the study of vibrations of stressed shells and their stability. In his work, Lomen was interested only in the first term of the expansion. Here, we shall derive also the second term of the series as well. We shall assume, however, that the thickness ratio of the membrane h/d is so much smaller than ϵ that the variations of stresses and deformations across the shell are negligible. Consequently, the displacements will be evaluated on the middle surface.

The metric properties of the membrane in its deformed state are first expanded. The covariant base vectors and metric tensor of the sphere are respectively denoted $\mathbf{a}_{,\alpha}$ and $a_{\alpha\beta}$. Also orthogonal curvilinear co-ordinates are chosen (e.g. usual spherical co-ordinates θ, ϕ); this ensures that $a_{12} = 0$. The position vector of a point on the deformed middle surface is expanded as

$$\mathbf{A} = \mathbf{a} + \epsilon \overset{(1)}{\mathbf{A}} + \epsilon^2 \overset{(2)}{\mathbf{A}} + O(\epsilon^3). \tag{4.8}$$

The normal vector becomes

$$\mathbf{N} = \mathbf{n} + \epsilon \overset{(1)}{\mathbf{N}} + O(\epsilon^2),$$

where

$$\overset{(1)}{\mathbf{N}} = \frac{\mathbf{a}_{,1} \wedge \overset{(1)}{\mathbf{A}}_{,2} + \overset{(1)}{\mathbf{A}}_{,1} \wedge \mathbf{a}_{,2}}{|\mathbf{a}_{,1} \wedge \overset{(1)}{\mathbf{A}}_{,2} + \overset{(1)}{\mathbf{A}}_{,1} \wedge \mathbf{a}_{,2}|}.$$

The metric tensor of the deformed surface is given by

$$A_{\alpha\beta} = a_{\alpha\beta} + \epsilon \overset{(1)}{A}_{\alpha\beta} + \epsilon^2 \overset{(2)}{A}_{\alpha\beta} + O(\epsilon^3),$$

where
$$a_{\alpha\beta} = \mathbf{a}_{,\alpha} \cdot \mathbf{a}_{,\beta},$$
$$\overset{(1)}{A}_{\alpha\beta} = \mathbf{a}_{,\alpha} \cdot \overset{(1)}{\mathbf{A}}_{,\beta} + \mathbf{a}_{,\beta} \cdot \overset{(1)}{\mathbf{A}}_{,\alpha}, \tag{4.9}$$
$$\overset{(2)}{A}_{\alpha\beta} = \mathbf{a}_{,\alpha} \cdot \overset{(2)}{\mathbf{A}}_{,\beta} + \mathbf{a}_{,\beta} \cdot \overset{(2)}{\mathbf{A}}_{,\alpha} + \overset{(1)}{\mathbf{A}}_{,\alpha} \cdot \overset{(1)}{\mathbf{A}}_{,\beta}. \tag{4.10}$$
Its determinant is denoted
$$A = a + \epsilon \overset{(1)}{A} + \epsilon^2 \overset{(2)}{A} + O(\epsilon^3),$$
where
$$a = a_{11} a_{22},$$
$$\overset{(1)}{A} = a_{22} \overset{(1)}{A}_{11} + a_{11} \overset{(1)}{A}_{22},$$
$$\overset{(2)}{A} = a_{22} \overset{(2)}{A}_{11} + a_{11} \overset{(2)}{A}_{22} + \det(\overset{(1)}{A}_{\alpha\beta}),$$
$$\det(\overset{(1)}{A}_{\alpha\beta}) = \overset{(1)}{A}_{11} \overset{(1)}{A}_{22} - \overset{(1)}{A}{}_{12}^2.$$
The contravariant components are obtained by means of definition (3.5) and are similarly denoted
$$A^{\alpha\beta} = a^{\alpha\beta} + \epsilon \overset{(1)}{A}{}^{\alpha\beta} + \epsilon^2 \overset{(2)}{A}{}^{\alpha\beta} + O(\epsilon^3).$$
The strain invariants are found to be
$$I_1 = 3 + \epsilon^2 \left[a^{\alpha\beta} \overset{(2)}{A}_{\alpha\beta} - \frac{\overset{(2)}{A}}{a} + \left(\frac{\overset{(1)}{A}}{a} \right)^2 \right] + O(\epsilon^3),$$
$$I_2 = 3 + \epsilon^2 \left[a_{\alpha\beta} \overset{(2)}{A}{}^{\alpha\beta} + \frac{\overset{(2)}{A}}{a} \right] + O(\epsilon^3),$$
$$I_3 = 1.$$
From the incompressibility requirement and from (3.6), the thickness ratio of the membrane is given by
$$k^{-2} = 1 + \epsilon \frac{\overset{(1)}{A}}{a} + \epsilon^2 \frac{\overset{(2)}{A}}{a} + O(\epsilon^3).$$
The material functions Φ, Ψ and P are expanded too and, from (3.11) and (3.12), they are related by
$$\overset{(0)}{\Phi} + 2\overset{(0)}{\Psi} + \overset{(0)}{P} = 0,$$
$$\overset{(1)}{\Phi} = \overset{(1)}{\Psi} = 0, \quad \overset{(1)}{P} = \frac{\overset{(1)}{A}}{a} (\overset{(0)}{\Phi} + \overset{(0)}{\Psi}),$$
$$\overset{(2)}{\Phi} + 2\overset{(2)}{\Psi} + \overset{(2)}{P} = \left[\frac{\overset{(2)}{A}}{a} - \left(\frac{\overset{(1)}{A}}{a} \right)^2 \right] (\overset{(0)}{\Phi} + \overset{(0)}{\Psi}) + \overset{(0)}{\Psi} \frac{\det(\overset{(1)}{A}_{\alpha\beta})}{a}.$$
Consequently, the expansion of the stress resultants becomes
$$\tau^{\alpha\beta} = \epsilon \overset{(0)}{\tau}{}^{\alpha\beta} + \epsilon^2 \overset{(1)}{\tau}{}^{\alpha\beta} + \dots,$$
$$\overset{(0)}{\tau}{}^{\alpha\beta} = (\overset{(0)}{\Phi} + \overset{(0)}{\Psi}) \left(-\overset{(1)}{A}{}^{\alpha\beta} + \frac{\overset{(1)}{A}}{a} a^{\alpha\beta} \right) \Big/ E. \tag{4.11}$$

As should be expected, the first-order term in the expansion of the theory of large deformations leads to the small-deformation approximation. Furthermore, to this order of approximation, all materials become neo-Hookean since the stress–strain relation, as given by (4.11), is linear and depends only on one material coefficient, namely $\overset{(0)}{\Phi}+\overset{(0)}{\Psi}$. When relation (4.11) is compared to the classical theory of small deformations of Hookean membranes, the relation between the sum $\overset{(0)}{\Phi}+\overset{(0)}{\Psi}$ and the Young's modulus is

$$E = 3(\overset{(0)}{\Phi}+\overset{(0)}{\Psi}).$$

It is this value of E which is used in the non-dimensionalization process. Consequently, in non-dimensional form, the stress–strain relation depends now to $O(\epsilon^2)$ on only one additional parameter:

$$\Psi' = \frac{\overset{(0)}{\Psi}}{\overset{(0)}{\Phi}+\overset{(0)}{\Psi}}.$$

Thus $\overset{(0)}{\tau}{}^{\alpha\beta}$ and $\overset{(1)}{\tau}{}^{\alpha\beta}$ become

$$3\overset{(0)}{\tau}{}^{\alpha\beta} = -\overset{(1)}{A}{}^{\alpha\beta}+\frac{\overset{(1)}{A}}{a}a^{\alpha\beta},$$

$$3\overset{(1)}{\tau}{}^{\alpha\beta} = -\overset{(2)}{A}{}^{\alpha\beta}+\frac{\overset{(2)}{A}}{a}a^{\alpha\beta}+a^{\alpha\beta}\left[\Psi'\frac{\det(\overset{(1)}{\mathrm{A}}_{\alpha\beta})}{a}-(1-\Psi')\left(\frac{\overset{(1)}{A}}{a}\right)^2\right]+(1+\Psi')\frac{\overset{(1)}{A}}{a}\overset{(1)}{A}{}^{\alpha\beta}-\frac{3\overset{(1)}{A}}{2a}\overset{(0)}{\tau}{}^{\alpha\beta}. \tag{4.12}$$

Finally, the equilibrium equations of the shell as given by (2.7), (2.8) and (2.9) are also expanded, so that the $O(1)$ equilibrium condition becomes

$$\frac{\partial\overset{(0)}{\tau}{}^{\alpha\beta}}{\partial y^\alpha}+\overset{(0)}{\Gamma}{}^{\alpha}_{\alpha\gamma}\overset{(0)}{\tau}{}^{\gamma\beta}+\overset{(0)}{\Gamma}{}^{\beta}_{\alpha\gamma}\overset{(0)}{\tau}{}^{\alpha\gamma}+\overset{(0)}{q}{}^{\beta} = 0, \tag{4.13}$$

$$\overset{(0)}{\tau}{}^{\alpha\beta}\overset{(0)}{B}_{\alpha\beta}+\overset{(0)}{q}{}^{3} = 0,$$

where $\overset{(0)}{\Gamma}{}^{\beta}_{\alpha\gamma}$ and $\overset{(0)}{B}_{\alpha\beta}$ are respectively the Christoffel symbol and the second fundamental form of the sphere. Similarly, to $O(\epsilon)$, this condition becomes

$$\frac{\partial\overset{(1)}{\tau}{}^{\alpha\beta}}{\partial y^\alpha}+\overset{(0)}{\Gamma}{}^{\alpha}_{\alpha\gamma}\overset{(1)}{\tau}{}^{\gamma\beta}+\overset{(0)}{\Gamma}{}^{\beta}_{\alpha\gamma}\overset{(1)}{\tau}{}^{\alpha\gamma}+\overset{(1)}{q}{}^{\beta}+\overset{(1)}{\Gamma}{}^{\alpha}_{\alpha\gamma}\overset{(0)}{\tau}{}^{\gamma\beta}+\overset{(1)}{\Gamma}{}^{\beta}_{\alpha\gamma}\overset{(0)}{\tau}{}^{\alpha\gamma} = 0,$$

$$\overset{(1)}{\tau}{}^{\alpha\beta}\overset{(0)}{B}_{\alpha\beta}+\overset{(1)}{q}{}^{3}+\overset{(0)}{\tau}{}^{\alpha\beta}\overset{(1)}{B}_{\alpha\beta} = 0, \tag{4.14}$$

where $\overset{(1)}{\Gamma}{}^{\gamma}_{\alpha\beta}$ and $\overset{(1)}{B}_{\alpha\beta}$ refer to the $O(\epsilon)$ deformed surface, and are determined from the solution of the $O(1)$ problem.

This completes the description of the perturbation procedure. The successive approximations to the solution are given in the following sections.

5. The solution of the $O(1)$ problem

The $O(1)$ fluid problem is completely defined by equations (4.1) to (4.3), together with boundary conditions (4.7) and (2.3). The deformation of the membrane is described by equations (4.9), (4.11) and (4.13). The curvilinear co-ordinates are the classical spherical co-ordinates shown on figure 1, where, to simplify notation, the latitude angle ϕ is used for co-ordinate y^1, and the azimuthal angle θ is used for y^2.

The solution of this problem has been given by Guerlet, Barthès-Biesel & Stoltz (1977) who computed the stress resultants in the membrane and the $O(1)$ displacement vector. Accordingly, the tensions in the membrane are given by

$$\overset{(0)}{\tau}{}^{11} = \tfrac{5}{2}\sin 2\theta, \quad \overset{(0)}{\tau}{}^{22} = -\tfrac{5}{2}\cot^2\phi \sin 2\theta, \quad \overset{(0)}{\tau}{}^{12} = \tfrac{5}{2}\cot\phi \cos 2\theta. \tag{5.1}$$

Furthermore a membrane point which was at position $\mathbf{x}$ at time t on the sphere is displaced to position $\mathbf{X}$,

$$\mathbf{X} = \mathbf{x} + \epsilon \overset{(1)}{\mathbf{u}} + O(\epsilon^2), \tag{5.2}$$

where $\overset{(1)}{\mathbf{u}}$ has components in the $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ frame given by

$$\overset{(1)}{u}_i = (J_{lm} - K_{lm}) \frac{x_l x_m}{r^2} x_i + K_{il}\, x_l; \tag{5.3}$$

here J_{lm} and K_{lm} are symmetric tensors whose only non-zero components are respectively

$$J_{12} = \tfrac{25}{4} \quad \text{and} \quad K_{12} = \tfrac{15}{4}. \tag{5.4}$$

Correspondingly the shape of the membrane becomes ellipsoidal and can be described by

$$r^2 = 1 + 2\epsilon J_{lm} x_l x_m + O(\epsilon^2)$$

or

$$r^2 = 1 + 25\epsilon x_1 x_2 + O(\epsilon^2). \tag{5.5}$$

The intersection of the ellipsoid with the x_1, x_2 plane is shown on figure 3. It appears that the particle is oriented at 45° with respect to the streamlines. This orientation is important because, in most experimental set-ups, the particle is observed along the x_2 axis. Consequently, it is the projection of the particle in the x_1, x_3 plane which is measured. In order to infer the actual deformation, it is of course necessary to know the angle of orientation. Another interesting feature of the $O(1)$ solution concerns the motion of the membrane around the steady ellipsoidal shape, which is explained by this model as the superposition of the rotational motion upon the spatially constant displacement.

This model, however, presents two main drawbacks. The first is that, because of the special geometry chosen here, a sphere, the internal viscosity of the particle does not enter the first-order approximation. Consequently, the dependency of the deformation on λ is lost. In order to be able to evaluate the role of λ, it is necessary to compute at least the next-order term of the expansion. The second drawback is the small range of validity of the first-order solution, which can be estimated in the following way. In the equation of the surface (5.5), the lateral displacement of a point is ignored since it is

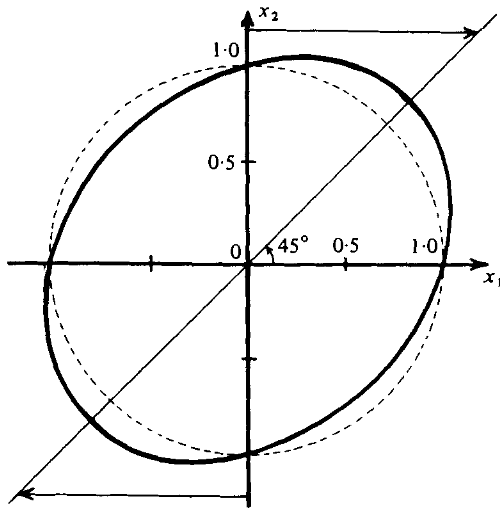

FIGURE 3. First-order deformation of the sphere, $\epsilon = 0{\cdot}02$. The elliptic curve represents the intersection of the ellipsoid with the x_1, x_2 plane. The dotted curve is the initial sphere.

$O(\epsilon^2)$. However, from the expression of $\overset{(1)}{u}_i$, given by (5.3), it appears that the maximum lateral displacement corresponds to an angular variation

$$\Delta\theta_{\max} = \tfrac{15}{4}\epsilon$$

so that ϵ cannot be much larger than 10^{-2}, if one wants to keep small the error made by truncating the power series after the $O(\epsilon)$ terms. For all these reasons, it was thought worth while to calculate the next-order term of the approximation, as will be presented in the next section.

However, before proceeding, it is necessary to extract some more information from the $O(1)$ solution. First of all, now that the deformed shape is known, all its metric properties (first and second fundamental forms, Christoffel symbols, etc.) can be determined. Their expression is given in the appendix. Second, the membrane velocity must be computed to $O(\epsilon)$, in order to be used afterwards as a boundary condition for the $O(\epsilon)$ fluid problem. This is done by superposing the rotational motion of the particle on the displacement of every point. Differentiating (5.2) with respect to time, we obtain

$$v_i^{(m)} = \frac{dX_i}{dt} = \frac{dx_i}{dt} + \epsilon\frac{d\overset{(1)}{u}_i}{dt} + O(\epsilon^2),$$

with

$$\frac{dx_i}{dt} = \overset{(0)}{v}{}_i^{(m)} = \Omega_{il}\,x_l.$$

The membrane velocity appears in the boundary conditions for the $O(\epsilon)$ problem. Consequently, $v_i^{(m)}$ must be expressed in terms of the Eulerian co-ordinates, X_i/r, of the membrane point in its deformed position:

$$v_i^{(m)} = \Omega_{il}\left[\frac{X_l}{r}\right]_{r=1} + \epsilon\left[\frac{d\overset{(1)}{u}_i}{dt} - \Omega_{il}\overset{(1)}{u}_l + \overset{(1)}{u}_p X_p\,\Omega_{il} X_l\right]_{r=1} + O(\epsilon^2)$$

or

$$v_i^{(m)} = \Omega_{il}\left[\frac{X_l}{r}\right]_{r=1} + \epsilon\,[J_{lm}\Omega_{ip}X_lX_mX_p + 2(J_{lm}-K_{lm})\,\Omega_{lp}X_mX_pX_i - \Omega_{il}K_{lm}X_m + K_{il}\Omega_{lp}X_p]_{r=1} + O(\epsilon^2). \quad (5.6)$$

It is now possible to derive the $O(\epsilon)$ term of the expansion.

6. The solution of the $O(\epsilon)$ problem

(*a*) *Motion of the fluids*

The $O(\epsilon)$ fluid problem is defined by the Stokes equations, with boundary conditions (4.4), (4.5) where $\overset{(1)}{v}{}_i^{(m)}$ is given by (5.6). Replacing $\overset{(0)}{v_i}$ and $\overset{(1)}{f}$ by their values obtained from § 5, the velocity boundary conditions for the external fluid motion become

$$\overset{(1)}{v_i} = 2(J_{lm}-K_{lm})\,\Omega_{lp}X_mX_pX_i - \Omega_{il}K_{lm}X_m + K_{il}\Omega_{lp}X_p - 5e_{ip}J_{lm}X_lX_mX_p + 5e_{pq}J_{lm}X_lX_mX_pX_qX_i,$$

$$\overset{(1)}{v_i}X_i = 2J_{kl}\Omega_{km}X_lX_m.$$

Similarly, the boundary conditions for the internal motion are

$$\overset{(1)}{v_i^*} = 2(J_{lm}-K_{lm})\,\Omega_{lp}X_mX_pX_i - \Omega_{il}K_{lm}X_m + K_{il}\Omega_{lp}X_p,$$

$$\overset{(1)}{v_i^*}X_i = 2J_{kl}\Omega_{km}X_lX_m.$$

The expressions for $\overset{(1)}{v_i}$ and $\overset{(1)}{v_i^*}$, both solutions of the Stokes equations, are given in terms of infinite series of spherical harmonics by Lamb (1932). The method of solution, explained in detail by Barthès-Biesel (1972), consists in integrating the equations over a unit sphere and in taking advantage of the orthogonality conditions. Consequently the spherical harmonics involved in $\overset{(1)}{v_i}$ and $\overset{(1)}{v_i^*}$ can be evaluated in terms of e_{ij}, J_{ij}, K_{ij} and Ω_{ij}. Similarly, from Lamb's solution, the stress tensors in the fluids are computed:

$$\begin{aligned}\overset{(1)}{\sigma_{ij}}X_j = {} & \tfrac{50}{7}Sd(J_{ik}e_{km})\,X_m + 4Sd(J_{ik}\Omega_{km})\,X_m - 10Sd(K_{ik}\Omega_{km})\,X_m \\ & - \tfrac{80}{7}Sd(J_{lk}e_{km})\,X_lX_mX_i - 16Sd(J_{lk}\Omega_{km})\,X_lX_mX_i + 16Sd(K_{lk}\Omega_{km})\,X_lX_mX_i \\ & - 20Sd_4(J_{lm}e_{pq})\,X_mX_lX_pX_qX_i + 15Sd_4(J_{il}e_{pq})\,X_lX_pX_q,\end{aligned}$$

and

$$\begin{aligned}\overset{(1)}{\sigma_{ij}^*}X_j = \lambda[& -(p^*/\lambda)\,X_i - 6Sd(J_{ik}\Omega_{kl})\,X_l + 10Sd(K_{ik}\Omega_{kl})\,X_l \\ & + 19Sd(J_{lk}\Omega_{km})\,X_lX_mX_i - 19Sd(K_{lk}\Omega_{km})\,X_lX_mX_i],\end{aligned}$$

where $Sd(A_{ij})$ and $Sd_4(A_{ijab})$ are deviators of order 2 and 4 respectively, which are symmetric with respect to any permutation of indices and have a zero contraction with respect to any two indices. Consequently they are defined by

$$Sd(A_{ij}) = \tfrac{1}{2}(A_{ij} + A_{ji} - \tfrac{2}{3}\delta_{ij}A_{ll}),$$

and

$$\begin{aligned}Sd_4(A_{ijab}) = \tfrac{1}{8}\{ & A_{ijab} + A_{iabj} + 22 \text{ other terms} \\ & - \tfrac{2}{7}[\delta_{ab}(A_{ijll} + A_{iljl} + 10 \text{ other terms}) + 5 \text{ other terms}] \\ & + \tfrac{8}{35}(\delta_{ij}\delta_{ab} + \delta_{ia}\delta_{bj} + \delta_{ib}\delta_{ja})\,(A_{llmm} + A_{lmlm} + A_{lmml})\}. \end{aligned} \quad (6.1)$$

The $O(\epsilon)$ load on the membrane, given by (4.6), becomes

$$\begin{aligned}\overset{(1)}{p}_i(X_k) &= p^* X_i + \tfrac{50}{7} Sd(J_{ik} e_{kl}) X_l + 2(2+3\lambda)\, Sd(J_{ik}\,\Omega_{kl}) X_l \\ &\quad - 10(1+\lambda)\, Sd(K_{ik}\,\Omega_{kl}) X_l - (19\lambda+16)\, Sd(J_{lk}\,\Omega_{km}) X_l X_m X_i \\ &\quad + (19\lambda+16)\, Sd(K_{lk}\,\Omega_{km}) X_l X_m X_i - \tfrac{80}{7} Sd(J_{mk} e_{kp}) X_m X_p X_i \\ &\quad + 15 Sd_4(J_{il} e_{pq}) X_l X_p X_q - 20 Sd_4(J_{km} e_{pq}) X_k X_m X_p X_q X_i \\ &\quad - 10 e_{pq} J_{il} X_p X_q X_l + 60 J_{lm} e_{pq} X_l X_m X_p X_q X_i \\ &\quad - 10 J_{lm} e_{pm} X_l X_p X_i - 25 e_{ip} J_{km} X_k X_m X_p, \end{aligned} \tag{6.2}$$

and the overall load on the membrane, at position X_k, is

$$p_i(X_k) = 5e_{ik}\left(\frac{X_k}{r}\right)_{r=1} + \epsilon \overset{(1)}{p}_i(X_k) + O(\epsilon^2).$$

It is this load which will determine the deformation of the shell. Consequently it should be expressed in terms of the initial co-ordinates x_k of the material point:

$$p_i(x_k) = 5(e_{ik}x_k)_{r=1} + \epsilon[\overset{(1)}{p}_i(x_k) + 5e_{ik}(\overset{(1)}{u}_k - \overset{(1)}{u}_l x_l x_k)]_{r=1} + O(\epsilon^2). \tag{6.3}$$

Now, the load is entirely defined by the above expressions, except for an unknown internal pressure p^*/λ, which will be determined later from the incompressibility condition. The value of each component $\overset{(1)}{p}_i$ can be readily obtained by replacing the tensors in (6.2) by their values given by (5.4) and by making use of definitions (6.1).

(b) Deformation of the membrane

The next step now consists in solving the equations of equilibrium (4·14) for the tensions in the membrane. However, the components of the viscous load must first be evaluated in the curvilinear co-ordinate system. The base vectors of this system have components in the $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ frame given by

$$\mathbf{A}_{,j} = A^i_j\,\mathbf{e}_i = (a^i_j + \epsilon \overset{(1)}{A^i_j})\,\mathbf{e}_i.$$

Consequently, the Cartesian and curvilinear components, respectively p^i and q^i, of the load are related by

$$(\overset{(0)}{p^i} + \epsilon \overset{(1)}{p^i})\,\mathbf{e}_i = (\overset{(0)}{q^j} + \epsilon \overset{(1)}{q^j})(a^i_j + \epsilon \overset{(1)}{A^i_j})\,\mathbf{e}_i, \tag{6.4}$$

where p^i and p_i are identical. If we introduce the tensor b^j_i such that

$$b^i_k a^k_j = \delta^i_j,$$

then

$$\overset{(0)}{q^j} = b^j_k \overset{(0)}{p^k}, \quad \overset{(1)}{q^j} = b^j_k \overset{(1)}{p^k} - b^j_k \overset{(1)}{A^k_l} b^l_m \overset{(0)}{p^m}. \tag{6.5}$$

Consequently, the $O(\epsilon)$ terms of the curvilinear components of the load are obtained from (6.5) where $\overset{(1)}{p^j}$ and $\overset{(0)}{p^j}$ correspond respectively to the $O(\epsilon)$ and $O(1)$ terms of (6.3),

$$\left.\begin{aligned} \overset{(1)}{q^1} &= \tfrac{25}{4}\sin\phi\cos\phi\,(-\tfrac{20}{7} + \tfrac{15}{4}\sin^2\phi - \tfrac{15}{4}\sin^2\phi\cos 4\theta + \cos 2\theta), \\ \overset{(1)}{q^2} &= \tfrac{25}{4}(-\sin 2\theta + \tfrac{15}{4}\sin^2\phi\sin 4\theta), \\ \overset{(1)}{q^3} &= \tfrac{25}{4}\,(\tfrac{4}{25}p^* + \tfrac{4}{7} - \tfrac{34}{7}\sin^2\phi + \tfrac{1}{5}(19\lambda+21)\sin^2\phi\cos 2\theta + \tfrac{7}{2}\sin^4\phi - \tfrac{7}{2}\sin^4\phi\cos 4\theta). \end{aligned}\right\} \tag{6.6}$$

To evaluate the $O(\epsilon)$ equilibrium equations (4.14), it is first necessary to compute the contribution arising from the $O(1)$ terms. This is done by using the values of $\overset{(0)}{\tau}{}^{\alpha\beta}$ given by (5.1) and the expressions for the Christoffel tensor of the deformed surface given in the appendix. Finally replacing the components of $\overset{(1)}{q}{}^{i}$ by their expressions (6.6), the equilibrium equations become

$$\frac{\partial \overset{(1)}{\tau}{}^{11}}{\partial\phi}+\frac{\partial \overset{(1)}{\tau}{}^{12}}{\partial\theta}+\cot\phi\,\overset{(1)}{\tau}{}^{11}-\cos\phi\sin\phi\,\overset{(1)}{\tau}{}^{22} = -\frac{25}{4}\sin\phi\cos\phi(\frac{15}{7}+\frac{5}{4}\sin^2\phi - \frac{5}{4}\sin^2\phi\cos 4\theta+\cos 2\theta), \tag{6.7}$$

$$\frac{\partial \overset{(1)}{\tau}{}^{12}}{\partial\phi}+\frac{\partial \overset{(1)}{\tau}{}^{22}}{\partial\theta}+3\cot\phi\,\overset{(1)}{\tau}{}^{12} = -\frac{25}{4}(-\sin 2\theta+\frac{5}{4}\sin^2\phi\sin 4\theta), \tag{6.8}$$

$$\overset{(1)}{\tau}{}^{11}+\sin^2\phi\,\overset{(1)}{\tau}{}^{22} = \tfrac{25}{4}\left[\frac{4}{25}p^{*}+\tfrac{32}{7}-\tfrac{62}{7}\sin^2\phi+\tfrac{5}{4}\sin^4\phi-\tfrac{5}{4}\sin^4\phi\cos 4\theta + \tfrac{1}{5}(19\lambda+21)\sin^2\phi\cos 2\theta\right]. \tag{6.9}$$

As should be expected the left-hand sides of (6.7), (6.8) and (6.9) are identical with those of the $O(1)$ analysis, corresponding to the small deformation theory. The method for solving this system of equations is classical and can be found in any textbooks on shell analysis. It consists in replacing $\overset{(1)}{\tau}{}^{22}$ in (6.7) and (6.8) by its value obtained from (6.9). Then the solution to the resulting system in $\overset{(1)}{\tau}{}^{11}$ and $\overset{(1)}{\tau}{}^{12}$ is sought in terms of Fourier series in θ, with coefficients depending on ϕ. One finds that

$$\overset{(1)}{\tau}{}^{11} = \frac{25}{4}\left[\frac{2}{25}p^{*}\frac{16}{7}-\frac{11}{4}\sin^2\phi+\cos 2\theta\left(\frac{19\lambda+26}{10}+\frac{19\lambda+16}{20}\sin^2\phi\right) - \frac{5}{4}\sin^2\phi\cos 4\theta\right],$$

$$\overset{(1)}{\tau}{}^{12} = \frac{25}{4}\cot\phi\left[-\frac{19\lambda+26}{10}\sin 2\theta+\frac{5}{4}\sin^2\phi\sin 4\theta\right],$$

$$\overset{(1)}{\tau}{}^{22} = \frac{25}{4\sin^2\phi}\left[\frac{2}{25}p^{*}+\frac{16}{7}-\frac{171}{28}\sin^2\phi+\frac{5}{4}\sin^4\phi - \cos 2\theta\left(\frac{19\lambda+26}{10}-\frac{57\lambda+68}{20}\sin^2\phi\right)+\frac{5}{4}\sin^2\phi\cos^2\phi\cos 4\theta\right].$$

Now equation (4.12) must be solved for the deformation tensor $\overset{(2)}{A}_{\alpha\beta}$, after replacing $\overset{(1)}{\tau}{}^{\alpha\beta}$ by the above values, and the $O(1)$ quantities by their expressions given in the appendix. It follows that:

$$\overset{(2)}{A}_{11} = 2\overset{(1)}{\tau}{}^{11}-a\overset{(1)}{\tau}{}^{22}+\frac{\overset{(1)}{A}}{2a}(2\overset{(0)}{\tau}{}^{11}-a\overset{(0)}{\tau}{}^{22})-\frac{\Psi'+2}{3a}\left[\det(\overset{(1)}{A}_{\alpha\beta})-\overset{(1)}{A}\left(2\overset{(1)}{A}_{11}-\frac{\overset{(1)}{A}_{22}}{a}\right)\right]+\frac{1-\Psi'}{3}\left(\frac{\overset{(1)}{A}}{a}\right)^2,$$

$$\overset{(2)}{A}_{22} = 2a\overset{(1)}{\tau}{}^{22}-\overset{(1)}{\tau}{}^{11}+\frac{\overset{(1)}{A}}{2a}(2a\overset{(0)}{\tau}{}^{22}-\overset{(0)}{\tau}{}^{11})-\frac{\Psi'+2}{3a}\left[\det(\overset{(1)}{A}_{\alpha\beta})-\overset{(1)}{A}\left(2\frac{\overset{(1)}{A}_{22}}{a}-\overset{(1)}{A}_{11}\right)\right]+\frac{1-\Psi'}{3}\left(\frac{\overset{(1)}{A}}{a}\right)^2,$$

$$\overset{(2)}{A}_{12} = 3a\overset{(1)}{\tau}{}^{12}+3\frac{\overset{(1)}{A}}{2}\overset{(0)}{\tau}{}^{12}+(\Psi'+2)\frac{\overset{(1)}{A}}{a}\overset{(1)}{A}_{12}.$$

The displacement components are then related to the metric tensor $\overset{(2)}{A}_{\alpha\beta}$ of the deformed surface. Following Lomen, the components of $\overset{(2)}{\mathbf{A}}$ are expressed in the curvilinear frame of the sphere:

$$\overset{(2)}{\mathbf{A}} = \overset{(2)}{w}_j \mathbf{a}^j = \overset{(2)}{u}_i \mathbf{e}_i,$$

where $\mathbf{a}^j$ are the contravariant base vectors of the sphere. Then, equation (4.10) becomes

$$\overset{(2)}{w}_{\alpha,\beta} + \overset{(2)}{w}_{\beta,\alpha} - 2\overset{(0)}{\Gamma}{}^{\gamma}_{\alpha\beta}\overset{(2)}{w}_{\gamma} - 2\overset{(0)}{B}_{\alpha\beta}\overset{(2)}{w}_3 = \overset{(2)}{A}_{\alpha\beta} - C_{\alpha\beta}, \tag{6.10}$$

where $C_{\alpha\beta} = \overset{(1)}{\mathbf{A}}_{\alpha} . \overset{(1)}{\mathbf{A}}_{\beta}$.

Equation (6.10) was derived by Lomen (1964), except for the $C_{\alpha\beta}$ term, which has to be added to take into account the higher order of the present analysis. In component form, (6.10) becomes

$$\left.\begin{aligned} 2\frac{\partial \overset{(2)}{w}_1}{\partial \phi} + 2\overset{(2)}{w}_3 &= \overset{(2)}{A}_{11} - C_{11}, \\ 2\frac{\partial \overset{(2)}{w}_2}{\partial \theta} + 2\cos\phi\sin\phi\, \overset{(2)}{w}_1 + 2\sin^2\phi\, \overset{(2)}{w}_3 &= \overset{(2)}{A}_{22} - C_{22}, \\ \frac{\partial \overset{(2)}{w}_1}{\partial \theta} + \frac{\partial \overset{(2)}{w}_2}{\partial \phi} - 2\cot\phi\, \overset{(2)}{w}_2 &= \overset{(2)}{A}_{12} - C_{12}. \end{aligned}\right\} \tag{6.11}$$

Again, as was noted earlier, the differential system (6.11) is similar to the small deformation theory system of equations. Consequently, the method of solution is well known. It consists, after elimination of $\overset{(2)}{w}_3$, of expanding the solution in terms of a Fourier series in θ, with coefficients depending on ϕ. One obtains

$$\begin{aligned} \overset{(2)}{w}_1 = \tfrac{25}{4}\cos\phi\sin\phi[-7{\cdot}91 - \tfrac{3}{4}\Psi'' - (0{\cdot}17 + \tfrac{1}{4}\Psi')\cos^2\phi + \tfrac{3}{20}(19\lambda + 26)\cos 2\theta \\ - (0{\cdot}17 + \tfrac{1}{4}\Psi')\sin^2\phi\cos 4\theta], \end{aligned} \tag{6.12}$$

$$\overset{(2)}{w}_2 = \tfrac{25}{4}\sin^2\phi[-\tfrac{3}{20}(19\lambda + 26)\sin 2\theta + (0{\cdot}17 + \tfrac{1}{4}\Psi')\sin^2\phi\sin 4\theta], \tag{6.13}$$

$$\begin{aligned} \overset{(2)}{w}_3 = \frac{25}{4}\Bigg[\frac{p^*}{25} - 5{\cdot}99 - \tfrac{1}{4}\Psi'' + \cos^2\phi(14{\cdot}27 + 2\Psi') \\ + \cos^4\phi\,(2{\cdot}67 + \tfrac{3}{4}\Psi') + \frac{209\lambda + 276}{40}\sin^2\phi\cos 2\theta \\ - (2{\cdot}67 + \tfrac{3}{4}\Psi')\sin^4\phi\cos 4\theta\Bigg]. \end{aligned} \tag{6.14}$$

(c) *Equation of the surface and internal pressure*

The relation between the initial and final positions of the material point is

$$\mathbf{X} = \mathbf{x} + \epsilon\overset{(1)}{\mathbf{u}} + \epsilon^2\overset{(2)}{\mathbf{u}} + O(\epsilon^2), \tag{6.15}$$

where $\overset{(1)}{\mathbf{u}}$ and $\overset{(2)}{\mathbf{u}}$ are respectively defined by (5.3) and by (6.12), (6.13) and (6.14). The distance of the point to the centre of the capsule is thus given by

$$r = (\mathbf{X}.\mathbf{X})^{\frac{1}{2}} = 1 + \epsilon\overset{(1)}{\mathbf{u}}.\mathbf{x} + \epsilon^2[\overset{(2)}{\mathbf{u}}.\mathbf{x} + \tfrac{1}{2}\overset{(1)}{\mathbf{u}}.\overset{(1)}{\mathbf{u}} - \tfrac{1}{2}(\overset{(1)}{\mathbf{u}}.\mathbf{x})^2] + O(\epsilon^3). \tag{6.16}$$

Consequently, from (6.15) and (6.16), with respect to the $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ frame

$$\frac{X_i}{r} = x_i + \epsilon(\overset{(1)}{u}_i - \overset{(1)}{u}_k x_k x_i) + \epsilon^2\left[\overset{(2)}{u}_i - \overset{(2)}{u}_k x_k x_i - \frac{x_i}{2}[\overset{(1)}{u}_k\overset{(1)}{u}_k - 3(\overset{(1)}{u}_k x_k)^2] - \overset{(1)}{u}_i\overset{(1)}{u}_k x_k\right] + O(\epsilon^3). \tag{6.17}$$

Equation (6.17) can be solved by successive approximations for x_i in terms of X_i/r. Replacing in (6.16) the resulting expression of x_i and the $O(1)$ tensors by their value, one obtains

$$r = 1 + \epsilon J_{km}\frac{X_k X_m}{r^2} + \epsilon^2\left[\overset{(2)}{w}_3 + (\tfrac{1}{2}K_{km}K_{mp} - 2J_{km}K_{mp})\frac{X_k X_p}{r^2} + (-\tfrac{1}{2}K_{km}K_{pq} + 2J_{km}K_{pq})\frac{X_k X_m X_p X_q}{r^4}\right] + O(\epsilon^3). \tag{6.18}$$

Here $\overset{(2)}{w}_3$ is given by (6.14), where the difference between the polar angles of the point before and after determination can be ignored. Evaluating expression (6.18) in terms of the spherical angles θ' and ϕ' corresponding to X_i, one finally obtains the polar equation of the surface of the membrane:

$$r = 1 + \tfrac{25}{4}\epsilon\sin^2\phi'\sin 2\theta' + \epsilon^2[\overset{(2)}{w}_3 - 19{\cdot}92(1 - \cos^4\phi' + \sin^4\phi'\cos 4\theta')] + O(\epsilon^3). \tag{6.19}$$

It is now possible to express the incompressibility of the microcapsule, and thus to infer the value of p^*. This is done by requiring the volume of the deformed capsule to be equal to $\frac{4}{3}\pi$:

$$V = \int_0^{2\pi}\int_0^{\pi}\frac{r^3}{3}\sin\phi'\,d\theta'\,d\phi' = \tfrac{4}{3}\pi.$$

Replacing r by its value (6.19), the $O(\epsilon^2)$ incompressibility condition becomes

$$\int_0^{2\pi}\int_0^{\pi}[\overset{(2)}{w}_3 - 19{\cdot}92(1 - \cos^4\phi' + \sin^4\phi'\cos 4\theta') + 39{\cdot}05\sin^4\phi'\sin^2 2\theta']\sin\phi'\,d\theta'\,d\phi' = 0.$$

After carrying out the integration, it follows that

$$\frac{p^*}{25} = 1{\cdot}58 - 0{\cdot}57\Psi''.$$

Consequently the final expression for $\overset{(2)}{w}_3$ becomes

$$\overset{(2)}{w}_3 = \frac{25}{4}\left\{-4{\cdot}41 - \tfrac{49}{60}\Psi'' + \cos^2\phi(14{\cdot}27 + 2\Psi'') + \cos^4\phi(2{\cdot}67 + \tfrac{3}{4}\Psi'') + \frac{209\lambda + 276}{40}\sin^2\phi\cos 2\theta - (2{\cdot}67 + \tfrac{3}{4}\Psi'')\sin^4\phi\cos 4\theta\right\}, \tag{6.20}$$

which completes the $O(\epsilon)$ problem.

7. Results and discussion

The equation of the surface of the microcapsule is given by (6.19) and (6.20). It depends on three parameters, namely ϵ, λ and Ψ'. As is apparent from (6.20), this analysis is limited to finite values of λ, since, to this order of approximation, $\overset{(2)}{w_3}$ is a monotonically increasing function of λ. Also, the elastic parameter Ψ' varies between 0 and 1. The value $\Psi' = 0$ corresponds to a neo-Hookean material, whereas the limit $\Psi' = 1$ corresponds to a Mooney material such that $C_1 = 0$ and $C_2 = 1$ (cf. equation (3.13)). The intersection of the surface with the $\mathbf{e}_1$, $\mathbf{e}_2$ plane is studied as a function of λ for a neo-Hookean material. On figure 4 the $O(\epsilon)$ and $O(\epsilon^2)$ profiles are compared. The $O(\epsilon^2)$ deformation of the surface is more pronounced than for the $O(\epsilon)$ ellipsoid. Two $O(\epsilon^2)$ profiles obtained for different values of λ are shown on figure 5. It is apparent that the orientation of the particle depends upon λ, the more viscous particles being more tilted towards the streamlines. Also, for large values of λ or of ϵ, the microcapsule presents a concave region, in the neighbourhood of which the membrane theory of shells no longer applies, and where bending moments as well as transverse shearing forces must be taken into account. Consequently the appearance of this concavity can be used to determine the range of validity of the approximation. Using this criterion, it is found that this limit is reached when ϵ is about 5×10^{-2}. This value may seem fairly small, but the relevant small parameter is 25ϵ rather than ϵ. The deformations attained when ϵ is of the order of 4×10^{-2} are already important. On figure 6 is shown the deformation of the particle, measured as the ratio $(L-B)/(L+B)$, where L and B are respectively the length and the breadth of the deformed capsule. It appears that, to this order of approximation, the elastic behaviour of the material has a relatively small influence on the deformation of the particle. For each profile, the angle $\theta'_{\max}$ corresponding to the largest value of r was computed. On figure 7, the orientation of the particle, as measured by $\theta'_{\max}$, is shown as a function of λ, thus confirming the fact that the viscous capsules tend to realign themselves with the streamlines. It should be noted that this behaviour is similar to that of liquid droplets. It is presently very difficult to compare the predictions of the model to direct experimental measurements of the deformation of such capsules. The only available data are related to the behaviour of red blood cells, but correspond to values of ϵ which are of order 1, and thus completely outside the range of validity of the analysis.

This model predicts that the membrane takes a rotational motion around the steady deformed shape of the capsule. A similar behaviour, called 'tank-treading', was first reported for red blood cells suspended in a simple shear flow by Schmid-Schönbein & Wells (1960). So far there has been considerable discussion regarding the exact mechanism of such tank-treading. The liquid droplet model has often been proposed, but recent experiments by Fischer, Stöhr-Liesen & Schmid-Schönbein (1978) show that the frequency of rotation of the interface is a linear function of shear rate and is independent of λ. This is of course in disagreement with the analysis of liquid droplets as given by Rumscheidt & Mason (1961). In the microcapsule model, on the contrary, the angular velocity of a membrane point is exactly equal to the shear rate for all λ's. Indeed the tank-treading motion results from the superposition of a rotation due to flow vorticity and of locally constant elastic deformations due to constant viscous forces. A comparison with low shear experimental results of Fischer & Schmid-Schönbein (1977) indicates that the theoretical rotational frequencies are about 1·5

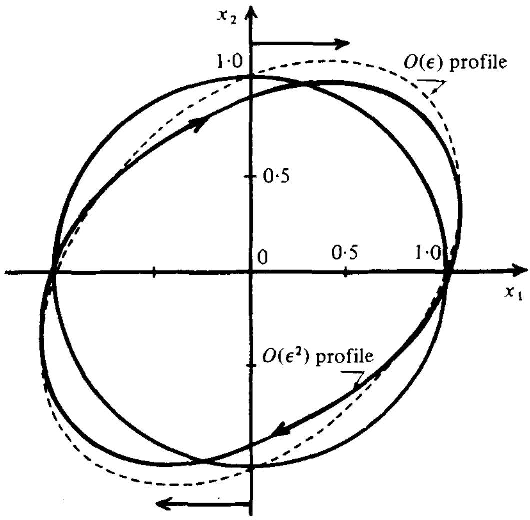

FIGURE 4. Deformation of the particle in the x_1, x_2 plane. $\lambda = 1$, $\epsilon = 0.03$.

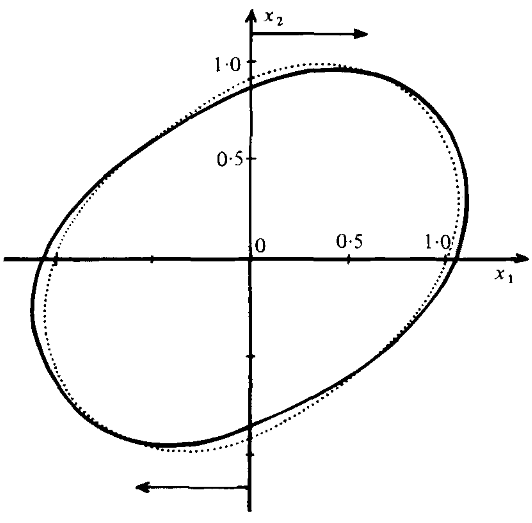

FIGURE 5. Deformation of the particle in the x_1, x_2 plane, as a function of λ, $\epsilon = 0{\cdot}03$. ......, $\lambda = 0$; ——, $\lambda = 2$.

times larger than the measured ones. The discrepancy can be attributed to the difference in geometry between the spherical microcapsule and the normal red blood cell, which has the shape of a biconcave disk in its undeformed state.

Obviously, the simple model proposed here cannot be expected to reproduce exactly all the striking properties of normal red blood cells. In particular, because of its spherical geometry, the movement of the microcapsule evolves continuously from solid body rotation to tank-treading motion. The red blood cell first behaves as a flexible solid disk at very low shear rates, and switches abruptly to tank-treading motion when the shear rate and the suspending fluid viscosity are above some critical values ($\lambda > 1$, $G > 0{\cdot}1$ N m^{-2} as shown by Goldsmith & Marlow (1972)).

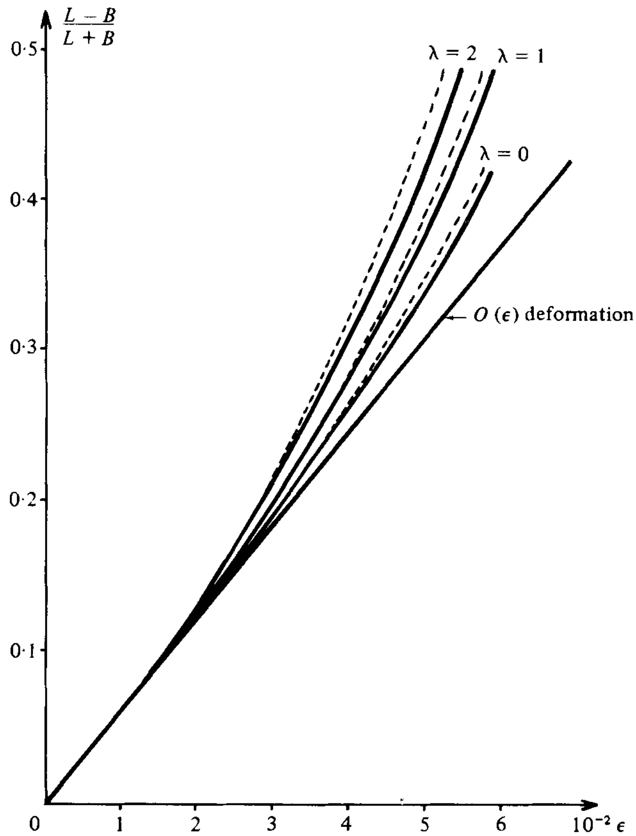

FIGURE 6. Deformation in the x_1, x_2 plane *versus* ϵ. The effect of the material behaviour of the membrane. ——, Neo-Hookean, $\Psi' = 0$; – – –, Mooney, $\Psi' = 1$.

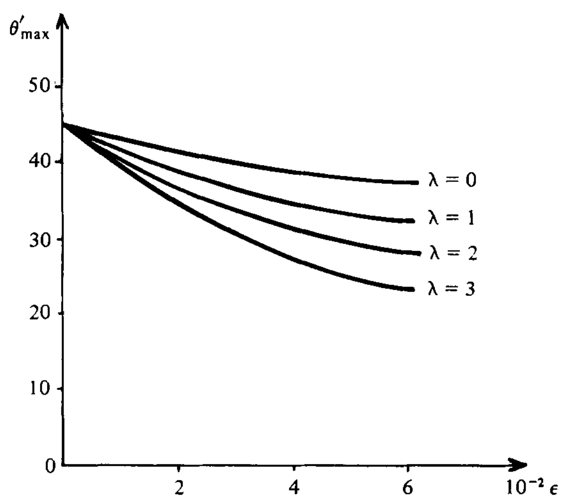

FIGURE 7. Orientation *versus* ϵ in the x_1, x_2 plane.

Nevertheless it is felt that this model explains in a simple fashion some aspects of the behaviour of human red blood cells suspended in simple shear flow.

In conclusion, it has been shown how the regular perturbation analysis of this formidable problem of microcapsule motion can be used to reach an approximate solution in the asymptotic case where the deformation remains limited. This has led to the derivation of an asymptotic development of the large deformation theory of membrane shells. Of course this analysis has a limited range of validity. However, it gives some interesting qualitative information on the behaviour of the microcapsule under shear. In particular it predicts the tank-treading motion of the membrane around the internal fluid contents. Furthermore, it indicates how the deformation and the orientation of the particle depend on physical properties such as its internal viscosity and its elastic properties. The same approach can be used to analyse the behaviour of such capsules when they are suspended in other types of flows. It should be interesting also to investigate the effect of viscoelastic properties of the membrane on the overall deformation of the particle. Finally this analysis can be viewed as the first step towards the solution of the complete problem in the general case where large deformations are considered.

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Appendix. Metric properties of the deformed microcapsule

The position vector of a point of the middle surface of the membrane is given to $O(\epsilon)$ by

$$\mathbf{A} = \mathbf{a} + \epsilon \overset{(1)}{\mathbf{A}} + O(\epsilon^2)$$

with
$$\mathbf{a} = \cos\theta \sin\phi\, \mathbf{e}_1 + \sin\theta \sin\phi\, \mathbf{e}_2 + \cos\phi\, \mathbf{e}_3,$$

$$\overset{(1)}{\mathbf{A}} = (\tfrac{5}{2}\sin^3\phi \sin 2\theta \cos\theta + \tfrac{15}{4}\sin\phi \sin\theta)\,\mathbf{e}_1 + (\tfrac{5}{2}\sin^3\phi \sin 2\theta \sin\theta + \tfrac{15}{4}\sin\phi\cos\theta)\,\mathbf{e}_2 + \tfrac{5}{2}\sin^2\phi \cos\phi \sin 2\theta\, \mathbf{e}_3.$$

The first fundamental form is defined by (3.4):

$$a_{11} = 1, \quad a_{22} = \sin^2\phi, \quad a_{12} = 0, \quad a = \sin^2\phi,$$

$$\overset{(1)}{A}_{11} = \tfrac{5}{2}\sin 2\theta(3 - \sin^2\phi), \quad \overset{(1)}{A}_{12} = \tfrac{15}{4}\cos 2\theta \sin 2\phi,$$

$$\overset{(1)}{A}_{22} = \tfrac{5}{2}\sin 2\theta \sin^2\phi(2\sin^2\phi - 3), \quad \frac{\overset{(1)}{A}}{a} = \tfrac{5}{2}\sin^2\phi \sin 2\theta,$$

$$\det(\overset{(1)}{A}_{\alpha\beta}) = -\tfrac{25}{4}\sin^2\phi(9\cos^2\phi + 2\sin^4\phi \sin^2 2\theta).$$

The Christoffel symbols are defined by (3.7).

For the sphere, the only non-zero components are

$$\overset{(0)}{\Gamma}{}^1_{22} = -\cos\phi \sin\phi, \quad \overset{(0)}{\Gamma}{}^2_{12} = \cot\phi.$$

For the ellipsoid the non-zero components are

$$\overset{(1)}{\Gamma}{}^1_{11} = -\tfrac{5}{4}\sin 2\phi \sin 2\theta,\quad \overset{(1)}{\Gamma}{}^2_{11} = -\tfrac{25}{2}\cos 2\theta,\quad \overset{(1)}{\Gamma}{}^2_{22} = -\tfrac{5}{2}\sin^2\phi \cos 2\theta,$$

$$\overset{(1)}{\Gamma}{}^1_{22} = -\tfrac{25}{2}\sin^3\phi \cos\phi \sin 2\theta,\quad \overset{(1)}{\Gamma}{}^1_{12} = 5\sin^2\phi \cos 2\theta,\quad \overset{(1)}{\Gamma}{}^2_{12} = \tfrac{5}{2}\sin 2\phi \sin 2\theta.$$

The second fundamental form, defined by (3.8), is given by

$$\overset{(0)}{B}_{11} = -1,\quad \overset{(0)}{B}_{22} = -\sin^2\phi,\quad \overset{(0)}{B}_{12} = 0,$$

and

$$\overset{(1)}{B}_{11} = \tfrac{5}{2}\sin 2\theta(2-\tfrac{13}{2}\sin^2\phi),\quad \overset{(1)}{B}_{12} = \tfrac{5}{2}\sin 2\phi \cos 2\theta,\quad \overset{(1)}{B}_{22} = -5\sin^2\phi \sin 2\theta(1+\tfrac{9}{4}\sin^2\phi).$$

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